MODEL ILLUSTRATING SOME PROPERTIES OF A HARDENING PLASTIC BODY

(MODEL', ILLIUSTRIRUIUSHCHAIA NEKOTORYE SVOISTVA UPROCHNAIUSHCHEGOSIA PLASTICHESKOGO TELA)

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Iu.N. RABOTNOV
(Moscow)

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In recent years many research papers have appeared on the problem of establishing the connection between the stresses and deformations in plastic bodies that have the hardening property. The existing experimental data have convincingly shown that under conditions of heavy loading neither the simplest theories of plasticity nor the theory of elasto-plastic deformations are applicable. To be able to explain experimental results, it has become necessary to introduce a number of new ideas, among them the concept of a corner or conical point on the loading surface. A recently published survey [1] relieves us of the necessity of presenting these new ideas here.

It should perhaps be pointed out that the hypotheses underlying the modern theory of plasticity are of a precise formal nature, while the experimental data are insufficiently definite and are ordinarily used only for indirect verification of the theory; they admit, as a rule, of various interpretations. This is why no one viewpoint prevails at present in regard to the direction of future development in the theory of plasticity.

The present author does not claim to have made progress in this direction; his aim is to illustrate with a simple example certain properties of an elasto-plastic body analogous to properties of hardening plastic material on the basis of realistic theories. It is known that properties of a hardening material can be modified by the use of a bar system, consisting of elasto-plastic elements of a material which does not have the property of hardening [2]. In the present paper this idea is developed for systems with two degrees of freedom.

Let us consider an elasto-plastic system subjected to the action of forces $Q_1, \ldots, Q_n$. Let the corresponding displacements be $q_1, \ldots, q_n$. In the n-dimensional space there exists the initial yield [or flow] surface $f(Q_1, \ldots, Q_n)$, which separates the region of elastic state from that of the elasto-plastic state. A simple loading is a loading along a ray starting at the origin of the coordinate system. It corresponds to a proportional increase of the forces. After a certain elasto-plastic state (represented by a point M of the n-dimensional space) is reached along some loading path, we can either continue the loading, which will be accompanied by plastic deformations, or we can unload the system. Hence, a surface can be passed through point M which is called the surface of plastic flow and which separates the region of plastic loading from the region of elastic loading.

Simple properties can be ascribed to the elements of this system, for example, it follows the diagram of uniaxial plasticity with an elastic unloading. The properties of the system as a whole, i.e. the relationship between the applied forces Q_i and the observed deformations q_i, can serve as an analogue of an elasto-plastic body.

It should be noted that the Batdorf-Budiansky [3,4] theory of plasticity is essentially based on consideration of the above model. Certain hypotheses expressed by these authors may give rise to objections (this is admitted even by the authors). For this reason we will start here with a simpler model made up of elements which cannot arouse any doubt as to their validity. As an example of such a model we select a thin pipe of elasto-plastic material which does not have the hardening property. This pipe is bent by two moments, M_x and M_y, in perpendicular planes.

Let us denote the radius of the pipe by R, its thickness by δ, the modulus of elasticity of the material by E, and the yield limit by σ_s. If M_x and M_y change proportionately, i.e. if we have a case of simple loading, then obviously there will occur simple bending of the pipe by a moment $M = \sqrt{M_x^2 + M_y^2}$. Without loss of generality, we can direct the x-axis along the neutral bending axis. The curvature of the bent axis of the pipe is given by

$$\varkappa = \frac{M}{\pi R^3 \delta E}$$

if the stresses do not exceed the yield limit.

If a plastic deformation takes place within the pipe, we will designate the polar angle of the boundary of the plastic zone by θ (Fig. 1). Since the deformation at the corresponding point is equal to $\sigma s/E$, we obtain

$$\varkappa R \sin\theta = \sigma_s / E$$

The stress σ, at the point determined by the polar angle a, is given by

$$\sigma = \sigma_s \frac{\sin \alpha}{\sin \theta} \tag{1}$$

Let us compute the moment produced by the stresses in the elastic part given by formula (1), and by the stresses σ_s in the plastic part. We then obtain

$$M = \pi R^2 \delta \sigma_s \frac{2\theta + \sin 2\theta}{\sin \theta}$$

We introduce the following dimensionless quantities:

$$Q = \frac{M}{R^2 \delta \sigma_s}, \qquad q = \frac{ER}{\sigma_s} \varkappa = \frac{1}{\sin \theta}$$

We obtain

$$Q = (2\theta + \sin 2\theta)\, q \qquad \text{in the elasto-plastic region}$$
$$\tag{2}$$
$$Q = \pi q \qquad \text{in the elastic region}$$

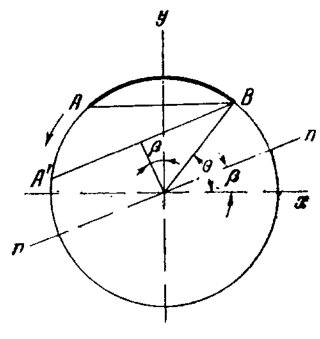

Fig. 1.

The quantity ($2\ \theta = \sin 2\ \theta$) depending on q can be called the plastic modulus, and we will denote if by E_s. The factor π in the second formula corresponds to the modulus of elasticity E_0. In Fig. 2 the relationship between Q and q is given.

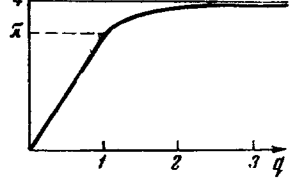

Fig. 2.

Returning to the general case when the moments M_k and M_y are acting, for the case of simple loading we now obtain the following relations:

$$Q_x = E_s q_x, \qquad Q_y = E_s q_y \tag{3}$$

The quantity E depends on $q = \sqrt{q_x^2 + q_y^2}$, or, $Q = \sqrt{Q_x^2 + Q_y^2}$. If $Q < \pi$, we must take $E_0 = \pi$ in formulas (2).

These relations are entirely analogous to the relations between stresses and deformations in the Nadai-Hencky theory.

The region of applicability of these relations is, however, not limited to simple loading, and the condition of unloading does not coincide with the Il'iushin condition. For further consideration it will be useful for us to have a geometric representation in a plane with coordinates Q_x, Q_y (Fig. 3). The circumference $Q = \pi$ here corresponds to the initial yield surface [or flow surface], and the path of simple loading corresponds to the ray passing through the origin of the coordinate system. The region of possible elasto-plastic states outside the initial yield surface also proves to be bounded. Indeed, the dimensionless moment cannot exceed the value $Q = 4$, which corresponds to the transition of the entire cross-section of the pipe into the plastic state. The circumference $Q = 4$ corresponds to the limiting yield surface. The concept of a limiting yield surface for a material with a limiting ability of hardening has not as yet been introduced into the theory of plasticity, though it appears to be a quite natural idea.

Let us now suppose that we have applied a simple loading up to a load value Q. Again, without restricting generality, we can assume that the loading has taken place along the Q_x-axis up to point M. The boundary of the plastic region is determined by the angle θ. We will continue the loading, changing not only Q_x but also Q_y, which was zero throughout the first stage of loading. The neutral axis of bending will no longer be the x-axis but some other line nn' (Fig. 1).

Thus, if no unloading occurs, formula (3) will remain valid. Hence there exist paths of loading which originate at point M and differ from the path of simple loading, but for which the relations of deformation theory remain valid. The extreme case is obtained when an increase in the plastic region occurs only from one side, for example the point A moves in the direction of the arrow to position A: while point B remains fixed. The straight line $A'B$ has to be parallel to the axis nn'. The bending moment relative to this axis can be evaluated by formula (2) if in place of the angle θ the angle $\theta' = \theta - \beta$ is substituted in it. In this way we get

$$Q_x = Q(\theta - \beta)\cos\beta, \qquad Q_y = Q(\theta - \beta)\sin\beta$$

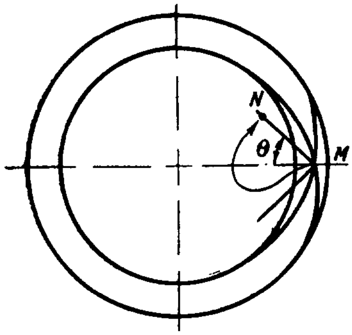

Fig. 3.

This means that the extreme path of loading (passing through point M), for which the relations of deformation theory are valid, can be obtained by rotating the curve, whose equation in polar coordinates is given by formula (2)

$$Q = Q(\theta)$$

about the origin of the coordinate system until it passes through point M. It is obvious that the second family of limiting curves can be obtained by rotating curve $Q = Q(-\theta)$.

The extreme paths of simple loading are shown in Fig. 3.

In the neighborhood of point M we have

$$\Delta Q_x = \frac{dQ}{d\theta}\cos\beta\,(-\Delta\beta) - Q\sin\beta\Delta\beta, \qquad \Delta Q_y = \frac{dQ}{d\theta}\sin\beta\,(-\Delta\beta) + Q\cos\beta\Delta\beta$$

but since β is small,

$$\Delta Q_x = -\frac{dQ}{d\theta}\Delta\beta, \qquad \Delta Q_y = Q\Delta\beta$$

We note that

$$\frac{dQ}{d\theta} = \frac{dQ}{dq}\frac{dq}{d\theta} = E_t Q \operatorname{ctg}\theta$$

Here

$$\frac{dQ}{dq} = 2\theta - \sin 2\theta = E_t$$

It is natural to call this quantity the tangent modulus. We can now determine the slope of the limiting paths of simple loading which pass through point M

$$\frac{\Delta Q_y}{\Delta Q_x} = \pm\frac{E_s}{E_t}\operatorname{tg}\theta \qquad (4)$$

The corresponding angle is equal to $\pi/2$ when $\theta = 0$, and $\theta = \pi/2$. Its minimum value of $76^{\circ}28'$ is attained when $\theta = 57^{\circ}30'$.

We will now obtain the conditions of elastic unloading after simple loading. It is obvious that the elastic unloading is here possible not only by a proportional decrease of the moments. The neutral axis of unloading can change position within certain limits. The extreme case will occur when this axis passes through the boundary of the plastic region. Let us suppose that a simple loading had been carried out along the x-axis up to a certain value of the dimensionless moment Q; the unloading has been realized in consequence of an additional bending relative to an axis making the angle ϕ with the x-axis. In consequence of the unloading, the moment relative to the x-axis changes by an amount Q_x' and a new moment Q_y' appears.

Owing to the fact that the unloading occurs in the elastic region, we may set

$$Q_x' = -\pi\omega\cos\varphi, \qquad Q_y' = -\pi\omega\sin\varphi$$

The corresponding stress is

$$\sigma' = -\omega\sigma_s\sin(\alpha-\varphi)$$

We will consider the unloading process along a path passing through point M. Then

$$Q_x = q - \pi\omega\cos\varphi, \qquad Q_y = -\pi\omega\sin\varphi \tag{5}$$

Thus the stresses are determined as follows:

$$\sigma = \sigma_s[1-\omega\sin(\alpha-\varphi)] \qquad (\theta \leqslant \alpha \leqslant \pi-\theta)$$

$$\sigma = \sigma_s\left[\frac{\sin\alpha}{\sin\theta} - \omega\sin(\alpha-\varphi)\right] \qquad \begin{pmatrix}0 \leqslant \varphi \leqslant \theta, \\ \pi-\theta, \leqslant \varphi \leqslant \pi\end{pmatrix} \tag{6}$$

The limiting path of unloading is given by the formulas (5) with $\phi = 0$. Therefore the curve, which in our model corresponds to the successive yield surface passing through M, has this point for a corner point. The limiting lines of unloading form an angle θ with the x-axis.

Comparing the results obtained under conditions for which the formulas of simple loading are valid, and the results obtained under conditions of unloading, we note that they agree qualitatively with the consequences of the Batdorf-Budiansky theory. According to this theory, the tangents drawn at the corner point bound the region of applicability both of the deformation theory and of the region of unloading, while in our theory the angle of unloading is found to be less than the angle of simple loading.

Let us next determine the boundaries of the region of unloading imposed by the demand that secondary plastic deformations can occur. For this purpose we must let $\sigma = -\sigma_s$ in formulas (6). In the plastic region we obtain:

$$1 - \omega \sin(\alpha - \varphi) = -1$$

When $\alpha - \phi = \pi/2$, we obtain $\omega = 2$. Thus, the boundary is the circle of radius 2π, with center at the point $x = Q$, $y = 0$. The investigation of the possibility of the occurrence of secondary plastic deformations in an original elastic region leads us to the discovery of a second boundary which is also a circle with center at the point $x = Q - \pi$ and of radius $\pi/\sin\theta$. In Fig. 4 the boundaries of the region of unloading (yield curves) for several values of θ are constructed.

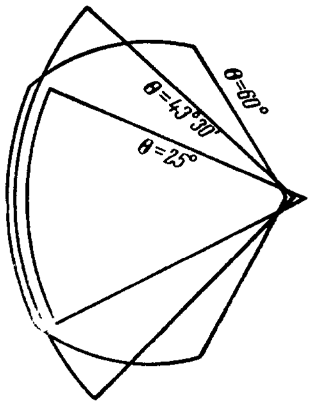

Fig. 4.

When θ is small, the elastic region is bound by straight lines and the arc of the first circle; when $\theta > 43^{\circ}30'$, it is bound by straight lines and arcs of two different circles.

For unloading paths originating at point *M* and not contained in the angles of simple loading and unloading, the analysis becomes more involved. We will limit ourselves to the consideration of small variations of the state of stress. Let us assume that the axis of the additional rotation makes (with the x-axis) an angle $\phi > \theta$, and therefore intersects the plastic zone. The additional stresses $\sigma' = \omega\sigma_s \sin(\alpha - \phi)$ occur only in the elastic region, i.e. when $-\phi < \alpha < \phi$ and $\pi - \phi < \alpha < \pi + \phi$. Computing the moments due to these stresses, we obtain the results in the form

$$\Delta Q_x = \Delta q_x \frac{E_t(\varphi) + E_t(0)}{2} + \Delta q_y (\sin^2\varphi - \sin^2 0)$$
$$\Delta Q_y = \Delta q_x (\sin^2\varphi - \sin^2 0) + \Delta q_y \frac{E_s(\varphi) + E_s(0)}{2} \qquad (7)$$

Here, obviously, $\tan\phi = \Delta q_y / \Delta q_x$. When $\phi = \theta$, we obtain from this the relations between the variations in the stresses and deformations which are analogous to those of the deformation theory of plasticity

$$\Delta Q_x = E_t \Delta q_x, \qquad \Delta Q_y = E_s \Delta q_y$$

When $\phi = \pi - \theta$, unloading occurs, and it follows from relation (7) that

$$\Delta Q_x = E_0 \Delta q_x, \qquad \Delta Q_y = E_0 \Delta q_y$$

The formulas (7) are quite complicated, and it is difficult to draw an analogy between their consequences and any one of the existing theories of plasticity. We can hardly, therefore, expect a simple description of the behavior of material under variable loads whose change of components deviates even slightly from proportionality. We note that if in the formulas (7) we separate the part corresponding to the plastic deformation, we then find that the increment vector of the plastic deformation makes an angle with the x-axis which does not exceed $\pi/2 - \theta$. From this it follows that our model satisfies the well-known Drucker condition

$$(Q_i - Q_i^*)\, \Delta q_i^p \geqslant 0$$

Here Q_i^* is the state represented by a point which lies inside the yield surface. We can easily verify that the relations (7) satisfy the fundamental Drucker definition of hardening:

$$\Delta Q_i\, \Delta q_i^p > 0$$

Finally we come to the last question to be considered here, which consists of the following. In the modern theory of plastic flow it is assumed that the function expressing the condition of plasticity coincides with the flow potential. This means that under an arbitrary change of the state of stress, the increment vector of the plastic deformation is directed along the normal to the yield [or flow] surface, if this surface is smooth. This follows directly from the above Drucker condition. As we have explained, the yield surface, in our model, consists of two rays passing through point M, and an arc of one of two circles. Let us assume that the system is loaded. After this we follow an arbitrary path of unloading lying entirely in the elastic region to a point of the straight line part of the yield curve, for example to point N (see Fig. 3). The state of stress corresponding to point N is such that the stress is everywhere less than the limit of plasticity and reaches this limit when $a = \theta$, decreasing as a linear function of the coordinates on at least one side of $a = \theta$. Next let us suppose that we have subjected the pipe to an additional bending with deformation components Δq_x and Δq_y. Over the entire cross-section of the pipe a change will take place in the stresses in accordance with Hokke's law, except that at the point $a = \theta$, where the stress was equal to the yield limit, this stress will remain unchanged. Around this point a small plastic region $\partial - \epsilon_1 \leqslant a \leqslant \theta + \epsilon_2$ will be formed. The same phenomenon occurs at the diametrically opposite point, of course.

In computing the moment we first of all extend the integration over the entire cross-section of the pipe, and obtain

$$\Delta Q_x' = E_0 \Delta q_x, \qquad \Delta Q_y' = E_0 \Delta q_y$$

After this we evaluate the moment from the stresses exceeding the yield limit which we have as it were added over the region $\theta - \epsilon_1 \leqslant a \leqslant \theta + \epsilon_2$. We denote this moment by Q_0, and point out that it has the components $Q_0 \sin\theta$ and $Q_0 \cos\theta$ along the coordinate axes. Thus,

$$\Delta Q_x = E_0 \Delta q_x - Q_0 \sin\theta, \qquad \Delta Q_y = E_0 \Delta q_y + Q_0 \cos\theta$$

Hence,

$$\Delta q_x = \frac{1}{E_0}\,\Delta Q_x + \frac{Q_0}{E_0}\sin\theta, \qquad \Delta q_y = \frac{1}{E_0}\,\Delta Q_y - \frac{Q_0}{E_0}\cos\theta$$

In these formulas, the first terms on the right-hand sides represent elastic deformations, while the second terms stand for plastic deformations. It can be seen that the increment vector of plastic deformation is perpendicular to the yield [or flow] surface.

In these considerations it is of course essential for quantities ϵ_1 and ϵ_2 to be infinitesimals of the same order as Δq_x and Δq_y. For this to be so, it is necessary for the distance NM to be large.

It is thus seen that the proposed simple model reproduces many properties of a plastic body which have been either revealed by experiments or postulated in some theories of plasticity, in particular the restricted applicability of relations among deformations, the presence of a corner point on the successive yield [or flow] surfaces when the initial yield surface is smooth, the Bauschinger effect, and the direction of the increment vector of plastic deformation along the normal to the yield surface wherever this surface is smooth. Moreover, the example presented reveals the great complexity which arises in any description of the processes of non-proportional loading.

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